New qualitative properties for viscosity solutions of fully nonlinear degenerate PDEs

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Geometric and functional inequalities and recent topics in nonlinear PDEs

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joint works with M. Bardi (Padova)

Strong Maximum Principle: the linear case

Let $\Omega \subset \mathbb{R}^d$ be open and connected and

$$Lu := -\mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u = 0,$$

A unif. ell., A, b bounded and continuous, c bounded and $c \geq 0$

Theorem (Hopf)

If $u \in C^2(\Omega)$, $c \geq 0$, $Lu \leq 0$ in Ω and u attains its greatest nonnegative value M at $x_0 \in \Omega$, then $u \equiv M$ in Ω.

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Another way to formulate the principle...

Problem

Let $x_0 \in \Omega$. Determine the largest connected, relatively closed subset $D(x_0)$ of Ω containing x_0 and such that

If $u \in C^2(\Omega)$, $Lu \leq 0$ in Ω and u attains its greatest nonnegative value M at $x_0 \in \Omega$, then $u \equiv M$ throughout $D(x_0)$. $D(x_0)$ = propagation set

It can be proved that the maximum propagates along a finite chain of trajectories of the diffusion vector fields

$$\pm X_j = \pm \sum_{k=1}^{d} a_{jk} \partial_k$$

In particular $\mathrm{rank}(\mathcal{L}(X_1, ..., X_d)) = d$ at $x \in \Omega \overset{\text{Chow's thm}}{\Longrightarrow} \exists U(x)$ such that any point of $U(x)$ can be joined to x by a finite chain of trajectories of the fields $\{X_j\}_{j=1}^{d}$.

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Theorem (Bony)

If the Lie algebra $\mathcal{L}(X_1, ..., X_d)$ has full rank at every point $x \in \Omega$, then (SMP) holds for L.

An example

Consider $\Omega = \{(x,y) : x \in (-1,1)\ , y \in (-1,1)\}$ and the linear operator

$$L = \partial_x^2 + x^2 \partial_y^2 = X^2 + Y^2$$

where

$$X = \partial_x\ , Y = x\partial_y\ .$$

We have that the vector fields X, Y and

$$[X, Y] = \partial_y$$

span $\mathbb{R}^2$ at all points of Ω, i.e. $\text{rank}\{\mathcal{L}(X, Y, [X, Y])\} = 2$ in the whole Ω. Then, for such (Grushin) operator, the (SMP) is valid.

Yet another characterization: subunit vector fields

- A vector $Z : \mathbb{R}^d \to \mathbb{R}$ is SUBUNIT in the sense of Fefferman-Phong for L if $A - Z \otimes Z \geq 0$, i.e.

$$a_{ij}(x)\xi_i\xi_j \geq |Z(x) \cdot \xi|^2 \ \forall \xi \in \mathbb{R}^d.$$

- Let $X_0 = \sum_{i=1}^d \left(b^i(x) - \sum_{j=1}^d \partial_j a_{ij}\right)\partial_i$ be the drift vector field of L.

Theorem (Taira)

The propagation set $D(x_0)$ of $x_0 \in \Omega$ contains the closure $\bar{D}(x_0)$ in Ω of all $y \in \Omega$ which can be joined to x_0 by a finite number of subunit and drift trajectories.

Remark. This characterization coincides with the probabilistic one by [Stroock-Varadhan] and those of [Hill,Redheffer,Bony], see [Taira].

Fully nonlinear eq.s: subunit vector fields revisited

Consider the fully nonlinear PDE

$$F(x, u(x), Du(x), D^2u(x)) = 0 \text{ in } \Omega \subset \mathbb{R}^d \ , F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathrm{Sym}_d \to \mathbb{R}$$

where F is proper, i.e. $F(x, r, p, X) \leq F(x, s, p, Y)$ for $r \leq s$, $Y \leq X$.

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Definition (Bardi-G.)

$Z \in \mathbb{R}^d$ is a generalized subunit vector field for F at $x \in \Omega$ if

$$\sup_\gamma F(x, 0, p, I - \gamma p \otimes p) > 0 \text{ for all } p \in \mathbb{R}^d \text{ s.t. } Z \cdot p \neq 0$$

Recall. When $F(x, D^2u) = -\mathrm{Tr}(A(x)D^2u)$, Fefferman-Phong's (FP) SV field for A is a vector Z such that $A \ge Z \otimes Z$.

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Lemma

Z is subunit in the sense of FP $\implies$ Z is generalized SV
If Z is generalized SV with F linear $\implies$ rZ with $r > 0$ small enough is subunit for A.

Some references on SMP for fully nonlinear PDEs

- [Caffarelli-Cabrè] strong minimum principle for $F(x, D^2u) = 0$, $F(x, 0) = 0$, F unif. ell. via weak Harnack inequality
- [Bardi-Da Lio] degenerate operators, concave and convex. Characterization of the propagation set, even probabilistic (see [Stroock-Varadhan] for the linear case)
- [Da Lio] for parabolic equations
- [Birindelli-Demengel et al] for fully nonlinear singular equations
- [Harvey-Lawson] for Hessian equations
- More recently, [Birindelli-Galise-Ishii] for truncated Laplacians

Propagation of maxima along generalized SVs

Assume that

$$\text{F is lower semicontinuous and proper} \tag{LSC+P}$$

$$F(x, \xi r, \xi p, \xi X) \geq \varphi(\xi) F(x, r, p, X)\ , \varphi > 0\ . \tag{SCAL}$$

Theorem (Bardi-G.)

Let F be such that (LSC+P)-(SCAL) hold, and assume it has a locally Lipschitz subunit vector field Z. Let $u \in USC(\Omega)$ be a viscosity subsolution to $F[u] = 0$ attaining a nonnegative maximum at $x_0 \in \Omega$. Then, $u(x) = u(x_0) = \max_\Omega u$ for all $x = y(s)$ for some $s \in \mathbb{R}$, where $y'(t) = Z(y(t))$ and $y(0) = x_0$.

Natural to consider the control system

$$y'(t) = \sum_{i=1}^{m} \Big(Z_i(y(t))\beta_i(t) \; , \beta_i \text{ measurable} \qquad (1)$$

If the system satisfies the property (BTC)

$\forall x_0, x_1 \in \Omega \; \exists$ trajectory of (1) with $y(0) = x_0, y(s) = x_1, \; y(t) \in \Omega \; \forall t \in [0, s$

then the maximum propagates on the whole Ω.

Recall. Hörmander condition for vector fields smooth enough $Z_1, ..., Z_m$ says that the vector fields and their commutators span $\mathbb{R}^d$ at every point of Ω.

Explicitly, this says that among the vector fields Z_i, their commutators $[Z_i, Z_j]$ and their iterated commutators, there exist d which are linearly independent.
In particular,
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Hörmander condition $\implies$ (BTC)

Theorem (Bardi-G.)

Let F be such that (LSC+P)-(SCAL) hold, and assume that there exist generalized subunit vector fields $Z_1, ..., Z_m$ for F satisfying the Hörmander condition. Then, any $u \in USC(\Omega)$ viscosity subsolution to $F[u] = 0$ attaining a nonnegative maximum at $x_0 \in \Omega$ is constant.

Remark. Similar scheme for the strong minimum principle, for $F \in USC$, proper and satisfying

$$F(x, \xi r, \xi p, \xi X) \leq \varphi(\xi) F(x, r, p, X)$$

Some examples: subelliptic PDEs

Given a family of smooth vector fields $X_1, ..., X_m$ one defines

$$\text{intrinsic (or horizontal) gradient } D_{\mathcal{X}}u = (X_1u, ..., X_mu)$$

$$\text{intrinsic (or horizontal) symmetrized Hessian } (D^2_{\mathcal{X}}u)^*_{ij} = \frac{X_iX_ju + X_jX_iu}{2}$$

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Given a family of smooth vector fields $X_1, ..., X_m$ one defines

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Then, natural to define

$$G[u] := G(x, u, \underbrace{D_\chi u}_{\in \mathbb{R}^m}, \underbrace{(D^2_\chi u)^*}_{\in \mathrm{Sym}_m}) = 0$$

See [Manfredi] lecture notes.

The Heisenberg group

In $\mathbb{R}^3$ consider (x, y, z) and the vector fields

$$X = \partial_x + 2y\partial_z \text{ and } Y = \partial_y - 2x\partial_z$$

Then $D_\chi u = (Xu, Yu)$, so $m = 2$ and $d = 3$ $(m \leq d)$. Write the coefficients of X, Y through

$$\sigma(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2y & -2x \end{pmatrix}$$

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$$\sigma(x) = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2y & -2x \end{pmatrix}\right)$$

Then $D_\chi u = \sigma^T Du$ and $(D_\chi^2 u)^* = \sigma^T(x) D^2 u \sigma(x)$, then

$$G(x, u, D_\chi u, (D_\chi^2 u)^*) = F(x, u, \sigma^T Du, \sigma^T(x) D^2 u \sigma(x))$$

Remark. Generally, $(D_\chi^2 u)^* = \sigma^T(x) D^2 u \sigma(x)$+first order terms. First order terms are null for step-2 Carnot groups (not for Engel group or Grushin).

Consider

$$G(x, u, D_\chi u, (D^2_\chi u)^*) = F(x, u, \sigma^T Du, \sigma^T(x) D^2 u \sigma(x) + g(x, Du))$$

Assume that G is elliptic in the following sense

$$\sup_{\gamma>0} G(x, 0, q, X - \gamma q \otimes q) > 0 \ \forall x \in \Omega \ , q \in \mathbb{R}^m \ , q \neq 0 \ , X \in \mathrm{Sym}_m \ . \quad (2)$$

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Lemma

If G satisfies (LSC+P)-(SCAL), then F fulfills the same properties. Moreover, the columns of σ are generalized subunit vector fields for F.

Corollary (Bardi-G.)

Under the standing assumptions, the (SMP) holds for the fully nonlinear subelliptic equation $G(x, u, D_\chi u, (D^2_\chi u)^*) = 0$

Some examples of PDEs

Pucci's extremal operators with ell. constants $0 < \lambda \leq \Lambda$ (see [Caffarelli-Cabrè])

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup\{-\mathrm{Tr}(AM) : \lambda I_d \leq A \leq \Lambda I_d\} = -\lambda \sum_{e_k>0} e_k - \Lambda \sum_{e_k<0} e_k$$

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf\{-\mathrm{Tr}(AM) : \lambda I_d \leq A \leq \Lambda I_d\} = -\Lambda \sum_{e_k<0} e_k - \lambda \sum_{e_k<0} e_k$$

Remark. Different from those introduced by [Pucci '66], where inf and sup are taken over a different class of matrices.

Important note. F is uniformly elliptic when

$$\lambda \mathrm{Tr}(Q) \leq F(x, r, p, M) - F(x, r, p, M + Q) \leq \Lambda \mathrm{Tr}(Q)$$

F unif. ell. if and only if

$$\mathcal{M}^-_{\lambda,\Lambda}(Q) \leq F(x, r, p, M) - F(x, r, p, Q) \leq \mathcal{M}^+_{\lambda,\Lambda}(Q) .$$

In general, our SMP is valid for

$$a(x)E(D_{\mathcal{X}}u,(D^2_{\mathcal{X}}u)^*)+c(x)|u|^{k-1}u=0$$

E hom. of degree $\alpha \leq k$ and

$$c \geq 0\ , a>0 \text{ and either } c=0 \text{ or } \alpha \leq k\ , k>0$$

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- subelliptic Pucci's operators over $(D^2_\chi u)^*$
- subelliptic ∞-Laplacian $-\Delta_{\chi,\infty}u = -D_\chi u(D^2_\chi u)^* D_\chi u$
- subelliptic p-Laplacian $-\Delta_{\chi,p}u = -\mathrm{div}_\chi(|D_\chi u|^{p-2}D_\chi u) = -(|D_\chi u|^{p-2}\Delta_\chi u + (p-2)|D_\chi u|^{p-4}\Delta_{\chi,\infty}u)$
- game theoretic p-Laplacian $\Delta^N_{\chi,p}u = -|D_\chi u|^{2-p}\mathrm{div}_\chi(|D_\chi u|^{p-2}D_\chi u)$
- subelliptic h-homogeneous ∞-Laplacian $-\Delta_{\chi,\infty}u = -|D_\chi u|^{h-3}\Delta_{\chi,\infty}u$

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- game theoretic p-Laplacian $\Delta^N_{\mathcal{X},p}u = -|D_{\mathcal{X}}u|^{2-p}\mathrm{div}_{\mathcal{X}}(|D_{\mathcal{X}}u|^{p-2}D_{\mathcal{X}}u)$
- subelliptic h-homogeneous ∞-Laplacian $-\Delta_{\mathcal{X},\infty}u = -|D_{\mathcal{X}}u|^{h-3}\Delta_{\mathcal{X},\infty}u$
- For $L^\alpha u = -\mathrm{Tr}(A^\alpha(x)D^2u) + b^\alpha(x) \cdot Du + c^\alpha(x)u$, consider the Bellman operators

  $$\sup_\alpha L^\alpha u \text{ or } \inf_\alpha L^\alpha u$$

  and Isaacs operators $\sup_\beta \inf_\alpha L^{\alpha,\beta}u$ or $\inf_\alpha \sup_\beta L^{\alpha,\beta}u$.

Some consequences: strong comparison principles

A well-known consequence of the (SMP) (actually an equivalent property) for linear equations is the

Theorem (Strong comparison principle (SCP))

Let Ω be a domain and $u, v \in C^2(\Omega)$ such that $Lu + c(x)u \leq Lv + c(x)v$. If $u \leq v$ in Ω and $u(x_0) = v(x_0)$ for some $x_0 \in \Omega$, then $u \equiv v$ in Ω.

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In the nonlinear case, it is not immediate to show that $w = u - v$ is a subsolution to $F[u] = 0$ and the properties are not equivalent. Actually, one has

$$\text{(SCP)} \implies \text{(SMP) provided that constants are solutions to the equation}$$

$$\text{(SMP)} \implies \text{(SCP) not necessarily true}$$

Somme comments on (SCP)

- Some minimal ellipticity is needed. For instance take $|u'| = 1$ on $(-1, 1)$ and note that $u_1(x) = x + 1$, $u_2(x) = -|x| + 1$ do not fulfill (SCP), see [Giga-Onhuma]
- Lipschitz growth in Du is important. In fact, one may take the viscous HJ

$$-\Delta u + |Du|^\gamma = 0 \ , \ \text{in } B(0, R) \ , \gamma \in (0, 1)$$

to observe that it has an explicit non-constant solution, cf [Barles-Diaz-Diaz]. Here, (SMinP) is valid, but neither (SMP) nor (SCP) hold.

Some references on (SCP)

- [Hopf] for classical solutions of fully nonlinear equations.
- [Trudinger] for Lipschitz continuous viscosity solutions.
- [Giga-Onhuma], [Ishii-Yoshimura] for some nonlinear second order elliptic equations.
- [Harvey-Lawson] for some Hessian equations
- [Patrizi] for fully nonlinear problems with Neumann boundary conditions
- [Pucci P.-Serrin] for some quasi-linear inequalities
- [Onhuma-Sakaguchi] for the prescribed mean curvature equation.

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General idea: find that $w = u - v$ is a subsolution to an equation $H(D^2 z) = 0$ which satisfies the (SMP)
Usually true for fully nonlinear uniformly elliptic operators depending on the Hessian, via the transitivity of viscosity inequalities, i.e. for F, G unif. ell.

$$F(M) + G(N) \geq H(M + N)\ , F(D^2 u) \leq f\ , G(D^2 v) \leq g\ , f, g \in C(\Omega)$$

$$\implies H(D^2(u + v)) \leq f + g$$

see [Caffarelli-Cabré].

A byproduct of our previous result is the following (SCP) for HJB eq.s

Theorem

Let

$$F_i[u] := \inf_{\alpha}\{-\mathrm{Tr}(A^\alpha(x)D^2u) + b^\alpha(x)\cdot Du + c^\alpha(x)u - f^\alpha(x)\} = 0\ , \quad (3)$$

$A^\alpha \geq 0$ and $c^\alpha \geq 0$. Assume the existence of subunit vector fields Z_i satisfying the Hörmander condition and such that $A^\alpha(x) \geq Z_i(x) \otimes Z_i(x)\ \forall \alpha, i, x$. If u, v are resp. a viscosity sub- and supersolution to (3), then (SCP) holds.

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Proof via a new comparison principle on small balls+(SMP)

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Proof via a new comparison principle on small balls+(SMP)
Remarkably, this implies the weak comparison when Ω is bounded in some new cases, e.g. when

$$-\mathrm{Tr}(A(x)(D^2_{\mathcal{X}}u)^*) + H(x, u, Du) = 0$$

One-side Liouville theorems

Degenerate elliptic equations on $\mathbb{R}^d$ of the general form

$$F(x,u,Du,D^2u) = 0 \text{ in } \mathbb{R}^d$$

Main questions:

- are subsolutions bounded from above constants?
- are supersolutions bounded from below constants?

Remark. Different issue from that of solutions, where the property is usually addressed via Harnack's inequality.

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Remark. Recall that for fully nonlinear degenerate equations over Hörmander vector fields, Harnack's inequality is, at this stage, unknown.

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However, different arguments allow to prove e.g. Hölder continuity in some degenerate (subelliptic) cases, see e.g. [Ferrari].

Liouville theorems via Harnack's inequality

Theorem (Liouville Theorem)

Let u be harmonic, i.e. $\Delta u = 0$, and $u \geq 0$ in $\mathbb{R}^d$. Then u is constant.

Proof Via the Harnack inequality $\sup_{B_R} u \leq C \inf_{B_R} u$, $C = C(d)$.

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Remark. A similar Liouville result can be deduced for SOLUTIONS to more general linear uniformly elliptic operators via suitable Harnack inequalities.

The case of sub- and supersolutions

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Theorem

(i) If u is subharmonic in $\mathbb{R}^2$ (i.e. solves $-\Delta u \leq 0$) and bounded from above then it is constant. (ii) For $\mathbb{R}^d$,$d \geq 3$, the Liouville property is false.

Proof. (i) Via Hadamard three-circle theorem (see next slides for a new proof).
(ii) Just take $u_1(x) = -(1 + |x|^2)^{-1/2}$ in $\mathbb{R}^3$ and $u_2(x) = -(1 + |x|^2)^{-1}$ in $\mathbb{R}^d$, $d \geq 4$.

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The case of sub and superharmonic functions is more delicate.

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Our goal: Use "Du (and u)" as good terms

The Liouville property in the subelliptic case

- One-side Liouville properties for solutions to $-\Delta_\chi u := \sum_{i=1}^m X_j^2 u = 0$ in Carnot groups $\mathbb{G}$ follows from Harnack's inequality [Bonfiglioli-Lanconelli-Uguzzoni] or mean-value formulas [Capuzzo Dolcetta-Cutrì].

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- False for sub-solutions to $-\Delta_{\chi}u \leq 0$ in $\mathbb{G}$. E.g. take $\mathbb{G} =$ Heisenberg group with $Q = 2d + 2$ hom. dim., and $\rho(x) = (|x_H|^4 + z^2)^{\frac{1}{4}}$

$$u_1(x) = \begin{cases} \frac{1}{8}[Q(Q-2)\rho^4 - 2(Q^2-4)\rho^2 + Q(Q+2)] & \text{if } \rho \leq 1 \ , \\ \frac{1}{\rho^{Q-2}} & \text{if } \rho \geq 1 \ , \end{cases}$$

- False even for Grushin geometries. For the Grushin plane, $X = \partial_x$, $Y = x\partial_y$, $\rho((x,y)) = (x^4 + 4y^2)^{\frac{1}{4}}$, consider

$$u_2(x) = \begin{cases} \frac{1}{8}[15 - 10\rho^2 + 3\rho^4] & \text{if } \rho \leq 1 \ , \\ 1/\rho & \text{if } \rho \geq 1 \ , \end{cases}$$

which solves $-\Delta_{\chi}u \leq 0$ in $\mathbb{R}^2$.

Non-divergent elliptic operators with drifts

Theorem

Let $u \in C^2$ be a solution to $Lu = -\mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du \leq 0$ in $\mathbb{R}^d$, with A unif. ell., A, b bounded and continuous. Suppose that there exists w supersolution to $Lw = 0$ for large x such that $\lim_{|x| \to \infty} w = +\infty$. If u is bounded by above, then it is constant.

Remark When $b = 0$ and $A = I_d$ we get $-\Delta u \leq 0$. Liouville property true for $d = 2$ taking $w(x) = \log |x|$.

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- Analogy with conditions to get Liouville properties on Riemannian manifolds, see [Grygoryan, Pucci P., Mari et al].
- Deep connection with stochastic properties, see [Grygoryan, Priola-Zabczyk].

Some references: Liouville properties

- [Caffarelli-Cabrè] for $F(x, D^2u) = 0$ with $F(x, 0) = 0$ via Harnack.
- [Cutrì-Leoni] $F = F(x, D^2u) + h(x)u^p$, F unif. ell. via Hadamard three-sphere thms
- [Capuzzo-Dolcetta-Cutrì] for $F = F(x, D^2u) + b(|x|)|Du| + h(x)u^p$, F unif. ell. using similar methods with b small at ∞, see also [Chen-Felmer].
- [Armstrong-Sirakov] generalize previous contributions via different methods.
- [Mannucci-Marchi-Tchou] for linear equations over Hörmander vector fields.
- [Capuzzo-Dolcetta-Cutrì, Birindelli-Capuzzo-Dolcetta-Cutrì] for semilinear PDEs on Carnot groups, [D'ambrosio-Lucente, Monticelli] on Grushin geometries.
- [Cutrì-Tchou] for Pucci's operators on the Heisenberg group.
- [Bordoni-Filippucci-Pucci] for quasi-linear PDEs on $\mathbb{H}^d$.
- [Bardi-Cesaroni] for fully nonlinear convex operators in $\mathbb{R}^d$.

Proof (Sketch). Step 1. $\xi > 0$ and $u_\xi(x) := u(x) - \xi w(x)$, with $C_\xi := \max_{|x|=\bar{R}} u_\xi(x)$
$\implies \lim_{|x|\to\infty} u_\xi(x) = -\infty$ and $\exists K > \bar{R}$ such that

$$u_\xi(x) \leq C_\xi \text{ for } |x| \geq K$$

Step 2. Trivially, $Lu_\xi = Lu - \xi Lw \leq 0$ for $|x| > \bar{R}$.

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Step 3. Apply the weak max principle on $\{x : \bar{R} < |x| < K\}$ to see that

$$\max_{\{x\in\mathbb{R}^d : \bar{R}\le|x|\le K\}} u_\xi(x) = \max_{\{x\in\mathbb{R}^d : |x|=\bar{R} \text{ or } |x|=K\}} u_\xi(x)$$

Since $u_\xi < C_\xi$ for $|x| \ge K$, we obtain by sending $\xi \to 0$

$$u(y) \le \max_{|x|=\bar{R}} u \text{ for } |y| > \bar{R}$$

By the weak max principle on $B_{\bar{R}}(0)$ we get

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Hence u attains its maximum at some point in $\partial B_{\bar{R}}(0)$.
Step 4. Apply the strong maximum principle to conclude that u is constant.

The general result

Consider

$$F(x, u, Du, D^2u) = 0 \text{ in } \mathbb{R}^d$$

Assume

1. F continuous, proper and satisfying $F[\varphi - \psi] \leq F[\varphi] - F[\psi]$, $\varphi, \psi \in C^2$ and $F(x, r, 0, 0) \geq 0$ for $r \geq 0$
2. F satisfies the comparison principle in any bounded open set Ω, namely if u and v are respectively a viscosity sub- and supersolution such that $u \leq v$ on $\partial\Omega$, then $u \leq v$ on Ω.
3. (Lyapunov function) There exists $R_0 \geq 0$ and $w \in LSC(\mathbb{R}^d)$ viscosity supersolution to $F[u] = 0$ outside $B(0, R_0)$ such that $\lim_{|x|\to\infty} w = +\infty$
4. F satisfies the (SMP).

Recall, (3) means that there exists $w \in LSC(\mathbb{R}^d)$ such that

$$F[w] \geq 0 \text{ for } |x| \geq R_0 \text{ such that } \lim_{|x|\to\infty} w = +\infty .$$

Theorem (Bardi-G.)

Assume (1)-(2)-(3) and (SCAL). Let $u \in USC(\mathbb{R}^d)$ be a viscosity subsolution to $F[u] = 0$ in $\mathbb{R}^d$ and

$$\limsup_{|x|\to\infty} \frac{u(x)}{w(x)} \leq 0$$

Then, u is constant.

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Then, u is constant.

The condition $\limsup_{|x|\to\infty} \frac{u(x)}{w(x)} \leq 0$ is satisfied if e.g. u is bounded from above.

Some motivations

- Large-time stabilization of solutions (in space) for degenerate parabolic equations

$$\partial_t u + G(x, D_\mathcal{X} u, (D^2_\mathcal{X} u)^*) = 0 \text{ in } \mathbb{R}^d \times [0, +\infty) \quad , u(x,0) = u_0(x) \text{ in } \mathbb{R}^d$$

  with $u_0 \in BUC(\mathbb{R}^d)$, see [Alvarez-Bardi, Bardi-Cesaroni]
- Ergodic problems for fully nonlinear subelliptic equations, see [Bardi-Cesaroni] for the fully nonlinear case, and [Marchi-Mannucci-Tchou] for some linear subelliptic cases.

Equations with horizontal gradient and Hessian

Consider

$$G(x, u, D_{\mathcal{X}} u, (D^2_{\mathcal{X}} u)^*) = 0$$

satisfying the following form of uniform sub-ellipticity

$$\mathcal{M}^-_{\lambda,\Lambda}(M - N) \leq G(x, r, p, M) - G(x, r, p, N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M - N)$$

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Take $N = 0$ and assume

$$G(x, r, p, 0) \ge H_i(x, r, p) = \inf_{\alpha \in A} \{c^\alpha(x) r - b^\alpha(x) \cdot p\}$$

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Then, if u is a subsolution to $G[u] = 0$ in $\mathbb{R}^d$, then u is a subsolution to

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}} u)^*) + \inf_{\alpha \in A} \{c^\alpha(x) u - b^\alpha(x) \cdot D_{\mathcal{X}} u\} = 0 \text{ in } \mathbb{R}^d$$

The Lyapunov function in the Heisenberg group

We need a function w which is a supersolution to

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We take $w(x) = \log \rho(x)$, ρ being the homogeneous norm of the underlying structure, and use that for Pucci's extremal operators one has

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}w)^*) = (-\Lambda(Q-1)+\lambda)|D_{\mathcal{X}}\rho|^2/\rho^2$$

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Similar procedure for equations driven by the normalized p-Laplacian over the Heisenberg vector fields

$$-\Delta^N_{p,\mathcal{X}}u = -\frac{1}{p}|D_{\mathcal{X}}u|^{2-p}\mathrm{div}_{\mathcal{X}}(|D_{\mathcal{X}}u|^{p-2}D_{\mathcal{X}}u)\ .$$

In fact, we have

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) \le -\Delta^N_{p,\mathcal{X}}u \le \mathcal{M}^+_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*)$$

for $\lambda = \min\left\{\frac{1}{p}, \frac{p-1}{p}\right\}, \Lambda = \min\left\{\frac{1}{p}, \frac{p-1}{p}\right\}$.

Theorem

If b^α is locally Lipschitz in x uniformly in α, $c^\alpha \geq 0$ continuous in $|x| \leq R$ uniformly in α. Assume for $\eta(x) = x_H|x_H|^2 + x_{2d+1}x_H^\perp$, $x_H^\perp = (x_{d+1}, ..., x_{2d}, -x_1, ..., -x_d)$

$$\sup_{\alpha\in A}\left\{b^\alpha(x)\cdot\frac{\eta}{|x_H|^2} - c^\alpha(x)\frac{\rho^4}{|x_H|^2}\log\rho\right\} \leq \lambda - \Lambda(Q-1)$$

If either $c^\alpha \equiv 0$ or $u \geq 0$, then (LP) holds.

The condition is satisfied for instance when

$$\limsup_{|x|\to\infty}\sup_\alpha b^\alpha(x)\cdot\frac{\eta}{|x_H|^2} < \lambda - \Lambda(Q-1)$$

since $c \geq 0$.

Some examples

- (Horizontal Ornstein-Uhlenbeck equation) Consider subsolutions of

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) - \gamma(x)\eta(x)\cdot D_{\mathcal{X}}u = 0 \text{ in } \mathbb{R}^{2d+1}$$

  where $\gamma(x) > 0$. Then, we have the Liouville property when

$$\liminf_{|x|\to\infty} \gamma(x)\rho^4(x) > \Lambda(Q-1) - \lambda$$

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Therefore, need γ large for large $|x|$, contrary to the results in [Capuzzo-Dolcetta et al]

- (Pucci-Schrödinger equation) For nonnegative subsolutions to

$$\mathcal{M}^-_{\lambda,\Lambda}((D^2_{\mathcal{X}}u)^*) + c(x)u = 0 \text{ in } \mathbb{R}^{2d+1}$$

the Liouville property holds when

$$\liminf_{|x|\to\infty} c(x)\frac{\rho^4(x)}{|x_H|^2}\log\rho(x) > \Lambda(Q-1) - \lambda$$

True if e.g. $c \sim |D_{\mathcal{X}}\rho|^2\rho^\gamma$ at infinity

Equations with Heisenberg Hessian and Euclidean gradient

Consider now

$$G(x,u,Du,(D^2_{\mathcal{X}}u)^*) = 0 \text{ in } \mathbb{R}^{2d+1}$$

Theorem

Under the same assumptions on b^α and c^α, suppose there exist $\gamma_1, ..., \gamma_{2d+1} \in \mathbb{R}$ with $\min_{\gamma_i} \gamma_1 = \gamma_0 > 0$ and such that

$$\sup_\alpha \{b^\alpha(x) \cdot D\rho(x)\} \le -\sum_{i=1}^{2d+1} \gamma_i x_i \partial_i \rho + o\left(\frac{1}{\rho^3}\right) \text{ as } \rho \to \infty$$

and

$$G(x,r,p,0) \ge H_i(x,r,p) = \inf_{\alpha \in A}\{c^\alpha(x)r - b^\alpha(x)\cdot p\}$$

If either $c^\alpha \equiv 0$ or $u \ge 0$, then (LP) holds.

Generalizations

- Quasi-linear subelliptic equations on Carnot groups and Grushin geometries.
- Fully nonlinear equations on H-type groups and Grushin plane.
- Case of fully nonlinear equations on the Heisenberg group via (SMP) and degenerate Hadamard three sphere theorems, following [Cutrì-Leoni,Cutrì-Tchou,Capuzzo Dolcetta-Cutrì], see [G.].
- (SMP) and (SCP) on Riemannian manifolds, see [G.-Pediconi]

Thanks for the attention!